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Some Important Properties of a New Type of Derivative

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Abstract: **In this paper, based on Riemann-Stieltjes integral, we study a new type of derivative. Some important properties of this new type of derivative is obtained, including derivative of inverse function, first derivative test, second derivative test, test of concavity, and L'Hôpital's rule. In fact, our results are generalizations of the results in classical differential calculus.**

Keywords: **Riemann-Stieltjes integral, new type of derivative, derivative of inverse function, first derivative test, second derivative test, test of concavity, L'Hôpital's rule.**

I. INTRODUCTION

The Riemann-Stieltjes integral, as a generalization of the Riemann integral, provides a more flexible integral method that makes the integral calculation more direct and convenient for specific types of functions. It has a wide range of applications in mathematical analysis, physics, financial mathematics, and economics, providing powerful tools for solving practical problems. With the development of mathematical analysis, research on the Riemann-Stieltjes integral remains active in the mathematical community, with new properties and applications continuously being discovered.

Based on the Riemann-Stieltjes integral, this paper studies a new type of derivative [1] and proves some important properties of this new type of derivative, such as derivative of inverse function, first derivative test, second derivative test, test of concavity, and L'Hôpital's rule. In fact, our results are generalizations of the results in traditional differential calculus. The theory of Riemann-Stieltjes integral can be referred to [2-3]. For books on calculus theory, we can refer to [4-5].

II. PRELIMINARIES

At first, we review the definition of Riemann-Stieltjes integral.

Definition 2.1 ([1])**:** Let $f, g: [a,b] \to R$. If the limit

$$\lim_{\|\Delta\|\to 0} \sum_{k=1}^{m} f(\xi_k)[g(x_k) - g(x_{k-1})]$$

exists, where $\Delta = \{a = x_0 < x_1 < \cdots < x_m = b\}$ is a partition of the interval $[a,b]$, $\xi_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$, and $\|\Delta\| = \max_{k=1,\cdots,m} \{\Delta x_k\}$. Then it is called the Riemann-Stieltjes integral of f with respect to g over $[a,b]$. We denote that

$$\lim_{\|\Delta\|\to 0} \sum_{k=1}^{m} f(\xi_k)\left(g(x_k) - g(x_{k-1})\right) = \int_a^b f(x)dg(x) = \int_a^b f\,dg, \quad (1)$$

and denote that $f \in R(g,[a,b])$. In particular, if $(x) = x$, then $\int_a^b f\,dg = \int_a^b f\,dx$, which is the Riemann integral of f on $[a,b]$.

Next, we present a new definition of derivative based on Riemann-Stieltjes integral.

Definition 2.2 ([1])**:** Let $x_0 \in (a,b)$ and $f(x), g(x)$ be functions defined on (a,b). If the limit

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)}$$

exists, then we say that f is differentiable with respect to g at x_0. If $f(x)$ are differentiable with respect to g at all $x \in (a,b)$, then f is said to be differentiable with respect to g on (a,b), and denoted by $f \in D(g,(a,b))$. In addition, the derivative of $f(x)$ with respect to g at x_0 is denoted by

$$f_g\ '(x_0) = \frac{d}{dg(x)} f(x)\Big|_{x=x_0} = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)}. \tag{2}$$

If $g(x) = x$, then $f_g\ '(x_0) = f'(x_0)$, which is the usual derivative of $f(x)$ at x_0. Moreover, for any positive integer n, we define

$$f_g^{(n)}(x_0) = \frac{d^n}{dg(x)^n} f(x)\Big|_{x=x_0} = \left(\frac{d}{dg(x)}\right)\left(\frac{d}{dg(x)}\right)\cdots\left(\frac{d}{dg(x)}\right) f(x)\Big|_{x=x_0}, \tag{3}$$

the n-th order derivative of $f(x)$ with respect to g at x_0.

Definition 2.3: For any $x_1, x_2 \in [a,b]$, $x_1 < x_2$. If $f(x_1) \le f(x_2)$, then we say that f is monotone increasing on $[a,b]$. If $f(x_1) \ge f(x_2)$, then f is monotone decreasing on $[a,b]$. In addition, if $f(x_1) < f(x_2)$, then f is strictly increasing on $[a,b]$. If $f(x_1) > f(x_2)$, then f is strictly decreasing on $[a,b]$.

Theorem 2.4 (Mean Value Theorem for Integrals) ([1])**:** *If g is a monotone increasing function on $[a,b]$, f is a continuous function on $[a,b]$, then there is $c \in [a,b]$ such that*

$$\int_a^b f(x)dg(x) = f(c)[g(b) - g(a)]. \tag{4}$$

Proposition 2.5: *Let k, C be real numbers, If $f, g, h: [a,b] \to R$ and f, h are differentiable with respect to g at $x_0 \in (a,b)$, then*

$$(f+h)_g{}'(x_0) = f_g\ '(x_0) + h_g\ '(x_0), \tag{5}$$

$$(f-h)_g{}'(x_0) = f_g\ '(x_0) - h_g\ '(x_0), \tag{6}$$

$$(kf)_g{}'(x_0) = kf_g\ '(x_0), \tag{7}$$

$$(C)_g{}' = 0. \tag{8}$$

Theorem 2.6: *If g is continuous at x_0, and f is differentiable with respect to g at x_0, then f is continuous at x_0.*

Theorem 2.7 (Product Rule) ([1])**:** *If g is continuous at x_0, and f, h are differentiable with respect to g at x_0, then $f \cdot h$ is differentiable with respect to g at x_0, and*

$$(f \cdot h)_g{}'(x_0) = f_g\ '(x_0) \cdot h(x_0) + f(x_0) \cdot h_g\ '(x_0). \tag{9}$$

Remark 2.8: In Theorem 2.7, it is easy to see that the condition ' g is continuous at x_0' can be replaced by 'function f or h is continuous at x_0' .

Theorem 2.9 (Quotient Rule) ([1])**:** *If function h is continuous at x_0, $h(x_0) \ne 0$, and f, h are differentiable with respect to g at x_0, then $\frac{f}{h}$ differentiable with respect to g at $\frac{f(x_0)}{h(x_0)}$, and*

$$\left(\frac{f}{h}\right)_g{}'(x_0) = \frac{f_g\ '(x_0) \cdot h(x_0) - f(x_0) \cdot h_g\ '(x_0)}{h^2(x_0)}. \tag{10}$$

Theorem 2.10 (Leibniz Rule) ([1])**:** *If p is a positive integer, function g is continuous at x_0, and f, h are p times differentiable with respect to g at x_0, then*

$$(f \cdot h)_g{}^{(p)}(x_0) = \sum_{k=0}^{p} \binom{p}{k} f_g^{(k)}(x_0) \cdot h_g{}^{(p-k)}(x_0), \tag{11}$$

where $\binom{p}{k} = \frac{p!}{k!(p-k)!}$.

Theorem 2.11 (Chain Rule) ([1])**:** *If the function h is continuous at x_0, h is differentiable with respect to g at x_0, and f is differentiable at $h(x_0)$, then the composite function $f \circ h$ is differentiable with respect to g at x_0, and*

$$(f \circ h)_g{}'(x_0) = f\ '(h(x_0)) \cdot h_g\ '(x_0). \tag{12}$$

Remark 2.12: In Theorem 2.11, the condition ' h is continuous at x_0' can be replaced by 'g is continuous at x_0'.

Theorem 2.13 (Mean Value Theorem for Derivatives) ([1])**:** *Let g be a strictly increasing function on $[a,b]$. If f is continuous on closed interval $[a,b]$ and differentiable with respect to g on open interval (a,b), then there exists $\xi \in (a,b)$ such that*

$$f(b) - f(a) = f_g\,'(\xi)[g(b) - g(a)]. \tag{13}$$

Theorem 2.14 (Cauchy's Mean Value Theorem)([1])**:** *Assume that g is a strictly increasing function on $[a,b]$. If f, h are continuous on $[a,b]$ and differentiable with respect to g on (a,b), $h(b) \neq h(a)$, and $h_g\,'(x) \neq 0$ for all $x \in (a,b)$. Then there is $\xi \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{h(b)-h(a)} = \frac{f_g\,'(\xi)}{h_g\,'(\xi)}. \tag{14}$$

Theorem 2.15 (Fundamental Theorem of Calculus) ([1])**:** *If g is a strictly increasing function on $[a,b]$, and f is continuous on $[a,b]$, then*

(I) *$G(x) = \int_a^x f(x)dg(x)$ is differentiable with respect to g on (a,b), and*

$$G_g\,'(x) = \frac{d}{dg(x)}\int_a^x f(x)dg(x) = f(x) \tag{15}$$

for all $x \in (a,b)$.

(II) *If $F(x)$ is continuous on $[a,b]$ and differentiable with respect to g on (a,b) with $F_g\,'(x) = f(x)$ for all $x \in (a,b)$, then*

$$\int_a^b f(x)dg(x) = F(b) - F(a). \tag{16}$$

III. MAIN RESULTS

In this section, we prove some important properties of this new type of derivative.

Theorem 3.1 (Derivative of Inverse Function): *Suppose that function g is differentiable at x_0 with $g\,'(x_0) \neq 0$, and f is an invertible function with the inverse f^{-1}. If f is differentiable at $f^{-1}(x_0)$ with $f\,'(f^{-1}(x_0)) \neq 0$, and if f^{-1} is continuous at x_0, and differentiable with respect to g at x_0, then f^{-1} is differentiable with respect to g at x_0, and*

$$f^{-1}{}_g\,'(x_0) = \frac{1}{f\,'(f^{-1}(x_0))\cdot g\,'(x_0)}. \tag{17}$$

Proof Since $f\left(f^{-1}(x)\right) = x$, it follows from chain rule that

$$f\,'\left(f^{-1}(x)\right)\cdot f^{-1}{}_g\,'(x) = \frac{1}{g\,'(x)}. \tag{18}$$

Thus,

$$f^{-1}{}_g\,'(x_0) = \frac{1}{f\,'(f^{-1}(x_0))\cdot g\,'(x_0)}.$$ q.e.d.

Theorem 3.2: *Assume that g is a strictly increasing function on $[a,b]$, and if f is continuous on $[a,b]$ and differentiable with respect to g on open interval (a,b).*

(I) *If $f_g\,'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on $[a,b]$.*

(II) *If $f_g\,'(x) < 0$ for all $x \in (a,b)$, then f is strictly decreasing on $[a,b]$.*

Proof (I) If $x_1, x_2 \in (a,b)$ and $x_1 < x_2$, then by mean value theorem for derivatives, there exists $\xi \in (a,b)$ such that

$$f(x_2) - f(x_1) = f_g\,'(\xi)[g(x_2) - g(x_1)]. \tag{19}$$

Since $f_g\,'(\xi) > 0$ and $g(x_2) - g(x_1) > 0$, it follows that $f(x_1) < f(x_2)$. Therefore, f is strictly increasing on (a,b). By the similar proof, we can obtain part (II) of this theorem. q.e.d.

Theorem 3.3 (First Derivative Test): *Suppose that* $x_0 \in (a,b)$, *and* g *is a strictly increasing function on* $[a,b]$. *If* f *is continuous on* $[a,b]$ *and differentiable with respect to* g *on* (a,b).

(I) *If* $f_g\ '(x)$ *changes from positive to negative at* x_0, *then* $f(x_0)$ *is a local maximum of* f.

(II) *If* $f_g\ '(x)$ *changes from negative to positive at* x_0, *then* $f(x_0)$ *is a local minimum of* f.

Proof (I) Since $f_g\ '(x)$ changes from positive to negative at x_0, there exists $a, b, a < b$ such that $f_g\ '(x) > 0$ for all $x \in (a, x_0)$ and $f_g\ '(x) < 0$ for all $x \in (x_0, b)$. By Theorem 3.2, f is strictly increasing on $[a, x_0]$ and strictly decreasing on $[x_0, b]$. Thus, $f(x_0)$ is a local maximum of f. Using the similar proof, we obtain part (II) of this theorem. q.e.d.

Theorem 3.4 (Second Derivative Test): *Let* $x_0 \in (a,b)$, *and* g *be a strictly increasing function on* $[a,b]$. *If* f *is continuous on* $[a,b]$ *and twice differentiable with respect to* g *on* (a,b), *and if* $f_g\ '(x_0) = 0$.

(I) *If* $f_g\ ''(x_0) > 0$, *then* $f(x_0)$ *is a local minimum of* f.

(II) *If* $f_g\ ''(x_0) < 0$, *then* $f(x_0)$ *is a local maximum of* f.

Proof (I) Since $f_g\ ''(x_0) = \lim_{x\to x_0} \frac{f_g\ '(x) - f_g\ '(x_0)}{g(x)-g(x_0)} = \lim_{x\to x_0} \frac{f_g\ '(x)}{g(x)-g(x_0)} > 0$, it follows that there is an open interval I containing x_0 for which $\frac{f_g\ '(x)}{g(x)-g(x_0)} > 0$ for all $x \neq x_0$ in I. If $x < x_0$, then $g(x) - g(x_0) < 0$, and hence $f_g\ '(x) < 0$. Also, if $x > x_0$, then $g(x) - g(x_0) > 0$ and hence $f_g\ '(x) > 0$. So, $f_g\ '(x)$ changes from negative to positive at x_0, and the first derivative test implies that $f(x_0)$ is a local minimum of f. Using the similar proof, we obtain part (II) of this theorem. q.e.d.

Definition 3.5 (Concavity): Let g be a strictly increasing and continuous function on $[a,b]$, and f be continuous on $[a,b]$ and differentiable with respect to g on (a,b). *If* $f_g\ '$ is strictly increasing on (a,b), then we say that f is concave upward with respect to g on (a,b). *If* $f_g\ '$ is strictly decreasing on (a,b), then f is concave downward with respect to g on (a,b).

Theorem 3.6 (Test for Concavity): *Suppose that* g *is a strictly increasing and continuous function on* $[a,b]$. *If* f *is continuous on* $[a,b]$ *and twice differentiable with respect to* g *on* (a,b).

(I) *If* $f_g\ ''(x) > 0$ *for all* $x \in (a,b)$, *then* f *is concave upward with respect to* g *on* (a,b).

(II) *If* $f_g\ ''(x) < 0$ *for all* $x \in (a,b)$, *then* f *is concave downward with respect to* g *on* (a,b).

Proof (I) For any $x_1, x_2 \in (a,b)$ and $x_1 < x_2$, by mean value theorem for derivatives, there is $\xi \in (x_1, x_2)$ such that

$$f_g\ '(x_2) - f_g\ '(x_1) = f_g\ ''(\xi)[g(x_2) - g(x_1)]. \tag{20}$$

Since g is strictly increasing on $[a,b]$ and $f_g\ ''(\xi) > 0$, it follows that $f_g\ '(x_1) < f_g\ '(x_2)$. By definition of concavity, f is concave upward with respect to g on (a,b). The similar proof can obtain part (II) of this theorem. q.e.d.

Theorem 3.7: *If* g *is strictly increasing on* $[a,b]$, f *is continuous on* $[a,b]$, *and* $u(x), v(x)$ *are continuous on* $[a,b]$ *and differentiable with respect to* g *on* (a,b). *Then* $H(x) = \int_{u(x)}^{v(x)} f(x)dg(x)$ *is differentiable with respect to* g *on* (a,b), *and*

$$H_g\ '(x) = \frac{d}{dg(x)} \int_{u(x)}^{v(x)} f(x)dg(x) = f(v(x)) \cdot v_g\ '(x) - f(u(x)) \cdot u_g\ '(x) \tag{21}$$

for all $x \in (a,b)$.

Proof Let $K(x) = \int_a^x f(x)dg(x)$, then

$$H(x) = \int_{u(x)}^{v(x)} f(x)dg(x) = -\int_a^{u(x)} f(x)dg(x) + \int_a^{v(x)} f(x)dg(x) = K(v(x)) - K(u(x)) . \tag{22}$$

By chain rule and fundamental theorem of calculus, we have

$$H_g\ '(x) = K'(v(x)) \cdot v_g\ '(x) - K'(u(x)) \cdot u_g\ '(x) = f(v(x)) \cdot v_g\ '(x) - f(u(x)) \cdot u_g\ '(x). \quad \text{q.e.d.}$$

Theorem 3.8 (L'Hôpital's Rule): *Let $x_0 \in (a,b)$ and g be a strictly increasing function on $[a,b]$. Suppose that f,h are continuous on $[a,b]$, and differentiable with respect to g on (a,b). If $\lim_{x\to x_0} f(x)=0$, $\lim_{x\to x_0} h(x)=0$, and that $\lim_{x\to x_0}\frac{f_g{}'(x)}{h_g{}'(x)}$ exists. Then*

$$\lim_{x\to x_0}\frac{f(x)}{h(x)}=\lim_{x\to x_0}\frac{f_g{}'(x)}{h_g{}'(x)}. \tag{23}$$

Proof Since $\lim_{x\to x_0}\frac{f_g{}'(x)}{h_g{}'(x)}$ exists, there is an interval I around x_0 (perhaps excluding x_0) where $f_g{}'(x)$ and $h_g{}'(x)$ exist and $h_g{}'(x)\neq 0$. Define two new functions F and H that agree with f and h for $x\neq x_0$, and set $F(x_0)=H(x_0)=0$. By Cauchy's mean value theorem applied to F and H, there is a point ξ between x_0 and x such that

$$\frac{f(x)}{h(x)}=\frac{F(x)}{H(x)}=\frac{F(x)-F(x_0)}{H(x)-H(x_0)}=\frac{F_g{}'(\xi)}{H_g{}'(\xi)}=\frac{f_g{}'(\xi)}{h_g{}'(\xi)}. \tag{24}$$

Since ξ between x_0 and x and $\lim_{x\to x_0}\frac{f_g{}'(x)}{h_g{}'(x)}$ exists, it follows that

$$\lim_{x\to x_0}\frac{f_g{}'(x)}{h_g{}'(x)}=\lim_{x\to x_0}\frac{f_g{}'(\xi)}{h_g{}'(\xi)}=\lim_{x\to x_0}\frac{f(x)}{h(x)}.$$ q.e.d.

IV. CONCLUSION

In this paper, we prove some important properties of a new type of derivative, including derivative of inverse function, first derivative test, second derivative test, test of concavity, and L'Hôpital's rule. In fact, our results are generalizations of ordinary differential calculus results. In the future, we will continue to use this new type of derivative to solve the problems in engineering mathematics and ordinary differential equations.

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